MATH 155/GRACEY NAME Key

EXAM 3/CHAPTER 10.1-10.5 HW________% E________% G________%

SHOW ALL WORK FOR FULL CREDIT/GIVE EXACT ANSWERS UNLESS OTHERWISE INDICATED

1. (22 POINTS) Consider the conic section $12x^2 + 20y^2 - 12x + 40y - 37 = 0$.

a. (6 POINTS) Write the equation in standard form.

$$12\left(x^2 - x + \left(\tfrac{1}{2}\right)^2\right) + 20\left(y^2 + 2y + (1)^2\right) = 37 + 3 + 20$$

$$\frac{12\left(x-\frac{1}{2}\right)^2}{60} + \frac{20(y+1)^2}{60} = \frac{60}{60}$$

$$\frac{\left(x-\frac{1}{2}\right)^2}{5} + \frac{(y+1)^2}{3} = 1$$

b. (8 POINTS) Analyze the equation. **If the equation does not have one of the following characteristics, write "none".**

i. (1 POINT) Find the vertex or the center.

$\boxed{\text{center: } \left(\tfrac{1}{2}, -1\right)}$

ii. (2 POINTS) Find the focus or foci.

$c^2 = a^2 - b^2$

$c^2 = 5 - 3$

$c = \sqrt{2}$

$\boxed{\text{Foci: } \left(\tfrac{1}{2} - \sqrt{2}, -1\right) \text{ and } \left(\tfrac{1}{2} + \sqrt{2}, -1\right)}$

iii. (1 POINT) Write the equation for the directrix.

$\boxed{\text{NONE}}$

iv. (4 POINTS) Write the equation for the asymptotes.

$\boxed{\text{NONE}}$

c. (6 POINTS) Find the zeros of the derivative with respect to x.

$\frac{d}{dx}\left(12x^2 + 20y^2 - 12x + 40y - 37\right) = \frac{d}{dx}(0)$

$24x + 40y\frac{dy}{dx} - 12 + 40\frac{dy}{dx} = 0$

$\frac{dy}{dx}(40y + 40) = 12 - 24x$

$\frac{dy}{dx} = \frac{\overset{3}{\cancel{12}}(1-2x)}{\underset{10}{\cancel{40}}(y+1)}$

$0 = \frac{3(1-2x)}{10(y+1)}$

$0 = 1 - 2x$

$2x = 1$

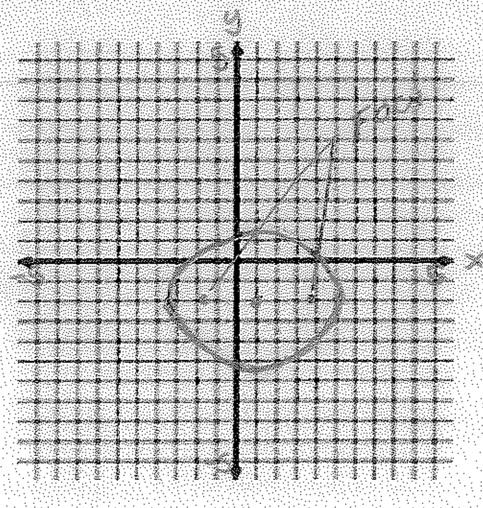
$x = \frac{1}{2}$

$\boxed{\left\{\tfrac{1}{2}\right\}}$

d. (2 POINTS) Sketch the graph by hand.

$a = \sqrt{5} \approx 2.2$

$b = \sqrt{3} \approx 1.7$



2. (6 POINTS) Find an equation of the hyperbola with

focus: $(20,0)$ and asymptotes: $y=\pm\frac{3}{4}x$

① $h=k=0$ so transverse axis must be vertical → $\frac{y^2}{a^2}-\frac{x^2}{b^2}=1$

② $a^2+b^2=20$ $\quad (\frac{3}{4}b)^2+b^2=20 \rightarrow b^2=20\cdot\frac{16}{25}$ $\quad$ and $\frac{a}{b}=\frac{3}{4}$ and $c^2=20$

$\frac{a}{b}=\frac{3}{4}$ $\quad \frac{9}{16}b^2+b^2=20$ $\quad b^2=\frac{64}{5}$ $\quad a^2=\frac{9}{16}b^2$ $\quad$ and $c^2=a^2+b^2$

$\rightarrow a=\frac{3}{4}b$ $\quad \frac{25}{16}b^2=20$ $\quad a=\frac{3}{4}b$ $\quad a^2=\frac{9}{16}(\frac{64}{5})$ $\quad a^2=36/5$

③ $\frac{y^2}{(\frac{36}{5})}-\frac{x^2}{(\frac{64}{5})}=1$ or $\frac{5y^2}{36}-\frac{5x^2}{64}$

3. (5 POINTS) Classify the graph of the equation as a circle, a parabola, an ellipse, or a hyperbola.

a. $25x^2-10x-200y-119=0$

$\boxed{\text{parabola}}$

b. $9(x+3)^2=36-4(y-2)^2$

$\boxed{\text{ellipse}}$

c. $9x^2+9y^2-36x+6y+34=0$

$\boxed{\text{circle}}$

d. $-x^2-x+y^2-5y=1+x^2$

$y^2-5y-2x^2-x=1$ $\quad \boxed{\text{hyperbola}}$

e. $2x^2-8x+y^2+5y=6$

$\boxed{\text{ellipse}}$

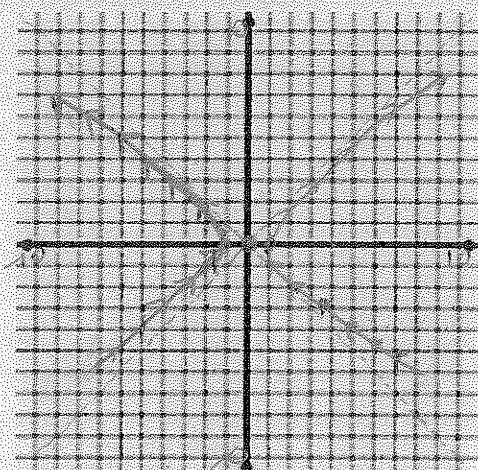
4. (16 POINTS) Consider the parametric equation $x=\sec\theta$ and $y=\tan\theta$.

a. (10 POINTS) Eliminate the parameter and graph the parametric equation by hand, indicating the orientation.

$\tan^2\theta+1=\sec^2\theta \rightarrow \sec^2\theta-\tan^2\theta=1$

$(x)^2-(y)^2=1$

$\frac{x^2}{1^2}-\frac{y^2}{1^2}=1$



orientation

| $\theta$ | $-\frac{\pi}{3}$ | $0$ | $\frac{\pi}{3}$ | $\frac{2\pi}{3}$ | $\pi$ | $\frac{4\pi}{3}$ |
|---|---|---|---|---|---|---|
| $x$ | 2 | 1 | 2 | $-2$ | $-1$ | $-2$ |
| $y$ | $-\sqrt{3}$ | 0 | $\sqrt{3}$ | $-\sqrt{3}$ | 0 | $\sqrt{3}$ |

b. (6 POINTS) Evaluate $\frac{dy}{dx}$.

$\frac{d}{dx}(x^2-y^2)=\frac{d}{dx}(1)$

$2x-2y\frac{dy}{dx}=0$

$\frac{dy}{dx}=\frac{-2x}{-2y}$

$\boxed{\frac{dy}{dx}=\frac{x}{y}}$

OR

$\frac{dy}{dx}=\frac{dy/d\theta}{dx/d\theta}$

$\frac{dy}{dx}=\frac{\sec^2\theta}{\sec\theta\tan\theta}$

$\frac{dy}{dx}=\sec\theta\cot\theta$

$\rightarrow \frac{dy}{dx}=\frac{1}{\cos\theta}\cdot\frac{\cos\theta}{\sin\theta}$

$\boxed{\frac{dy}{dx}=\csc\theta}$

5. (12 POINTS) Find the arc length of the curve

$x = \arcsin t$ and $y = \ln\sqrt{1-t^2}$ on the interval $0 \le t \le \frac{1}{2}$.

$y = \frac{1}{2}\ln(1-t^2)$

$\frac{dx}{dt} = \frac{1}{\sqrt{1-t^2}}$ , $\frac{dy}{dt} = \frac{-2t}{2(1-t^2)}$

$\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{\frac{1}{1-t^2} + \frac{t^2}{(1-t^2)^2}}$

$= \sqrt{\frac{1-t^2+t^2}{(1-t^2)^2}}$

$\sqrt{\frac{1}{(1-t^2)^2}}$

$= \frac{1}{1-t^2}$

$S = \int_0^{1/2} \frac{dt}{1-t^2}$

$S = \ln\left|\frac{1+t}{\sqrt{1-t^2}}\right|\Big|_0^{1/2}$

$S = \ln\frac{3/2}{\sqrt{1-(\frac{1}{2})^2}} - \ln\frac{1}{1}$

$S = \ln\frac{3/2}{\sqrt{1-\frac{1}{4}}} - 0$

$S = \ln\frac{3/2}{\sqrt{3}/2}$

$\boxed{S = \ln\sqrt{3} \text{ units}}$

trig sub.

$t = 1\sin\theta$
$dt = \cos\theta\, d\theta$
$1-t^2 = \cos^2\theta$

$\int \frac{dt}{1-t^2} = \int \frac{\cos\theta\, d\theta}{\cos^2\theta}$

$= \int \sec\theta\, d\theta$

$= \ln|\sec\theta + \tan\theta|$

$= \ln\left|\frac{1}{\sqrt{1-t^2}} + \frac{t}{\sqrt{1-t^2}}\right|$

$= \ln\left|\frac{1+t}{\sqrt{1-t^2}}\right|$

6. (4 POINTS) Find two sets of polar coordinates for the rectangular coordinate $\left(-3, \sqrt{3}\right)$.

$\left(2\sqrt{3}, \frac{5\pi}{6}\right)$ and $\left(2\sqrt{3}, \frac{17\pi}{6}\right)$

or $\left(-2\sqrt{3}, -\frac{\pi}{6}\right)$

$x = -3,\ y = \sqrt{3} \rightarrow$ Q II

$r = \sqrt{(-3)^2 + (\sqrt{3})^2}$ $\quad \tan\theta = \frac{\sqrt{3}}{-3}$

$r = \sqrt{12}$ $\quad \tan\theta = -\frac{1/2}{\sqrt{3}/2}$

$r = 2\sqrt{3}$ $\quad \theta = \frac{5\pi}{6}$

7. (10 POINTS) Find the area of the region inside $r = 3\sin\theta$ and outside $r = 1 + \sin\theta$.

Intersection points

$3\sin\theta = 1 + \sin\theta$

$2\sin\theta = 1$

$\sin\theta = \frac{1}{2}$

$\theta = \frac{\pi}{6}, \frac{5\pi}{6}$

$A = 2\left[\frac{1}{2}\int_{\pi/6}^{\pi/2} ((3\sin\theta - (1+\sin\theta))^2 d\theta\right]$

$A = \int_{\pi/6}^{\pi/2} (2\sin\theta - 1)^2 d\theta$

$A = \int_{\pi/6}^{\pi/2} (4\sin^2\theta - 4\sin\theta + 1) d\theta$

$A = 2\int_{\pi/6}^{\pi/2} (1 - \cos 2\theta) d\theta + (4\cos\theta + \theta\Big|_{\pi/6}^{\pi/2}$

$A = 2\left(\theta - \frac{\sin 2\theta}{2}\right)\Big|_{\pi/6}^{\pi/2} + \left[\left(4(0) + \frac{\pi}{2}\right) - \left(4\left(\frac{\sqrt{3}}{2}\right) + \frac{\pi}{6}\right)\right]$

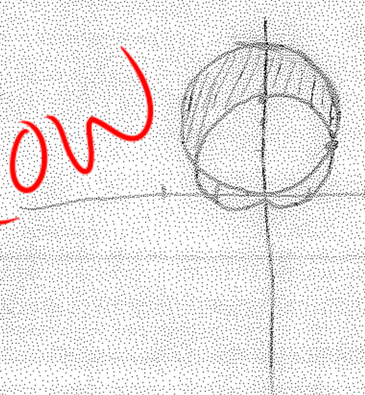
$A = 2\left[\left(\frac{\pi}{2} - 0\right) - \left(\frac{\pi}{6} - \left(\frac{\sqrt{3}/2}{2}\right)\right)\right] + \frac{\pi}{3} - 2\sqrt{3}$

$\rightarrow A = 2\left(\frac{\pi}{3} + \frac{\sqrt{3}}{4}\right) + \frac{\pi}{3} - 2\sqrt{3}$

$A = \frac{2\pi}{3} + \frac{\sqrt{3}}{2} + \frac{\pi}{3} - 2\sqrt{3}$

$A = \pi - \frac{3\sqrt{3}}{2}$

$\boxed{A = \frac{1}{2}(2\pi - 3\sqrt{3}) \text{ sq. units}}$



See Below

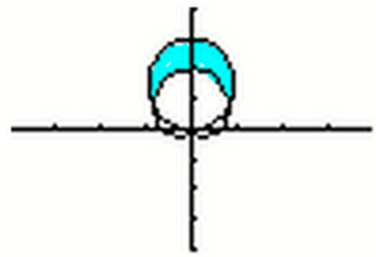
Ch.10 exam #7

Find area

inside $r = 3\sin\theta$

and

outside $r = 1+\sin\theta$



Limits of integration:

$3\sin\theta = 1+\sin\theta$

$2\sin\theta - 1 = 0$

$2\sin\theta = 1$

$\sin\theta = \frac{1}{2}$

$\theta = \frac{\pi}{6}, \frac{5\pi}{6}$

$$A = 2\cdot\frac{1}{2}\int_{\pi/6}^{\pi/2}\left[(3\sin\theta)^2 - (1+\sin\theta)^2\right]d\theta$$

$$A = \int_{\pi/6}^{\pi/2}\left(9\sin^2\theta - 1 - 2\sin\theta - \sin^2\theta\right)d\theta$$

$$A = \int_{\pi/6}^{\pi/2}\left(8\sin^2\theta - 2\sin\theta - 1\right)d\theta$$

$$A = 4\int_{\pi/6}^{\pi/2}(1-\cos 2\theta)\,d\theta + \left(2\cos\theta - \theta\right|_{\pi/6}^{\pi/2}$$

$$A = 4\left(\theta - \frac{\sin 2\theta}{2}\right|_{\pi/6}^{\pi/2} + \left[\left(2(0) - \frac{\pi}{2}\right) - \left(2\left(\frac{\sqrt{3}}{2}\right) - \frac{\pi}{6}\right)\right]$$

$$A = 4\left[\left(\pi/2 - 0\right) - \left(\pi/6 - \frac{\sqrt{3}/2}{2}\right)\right] + -\frac{\pi}{2} - \sqrt{3} + \frac{\pi}{6}$$

$\pi/3$

$$A = 4\left(\pi/2 - \pi/6 + \frac{\sqrt{3}}{4}\right) - \frac{\pi}{3} - \sqrt{3}$$

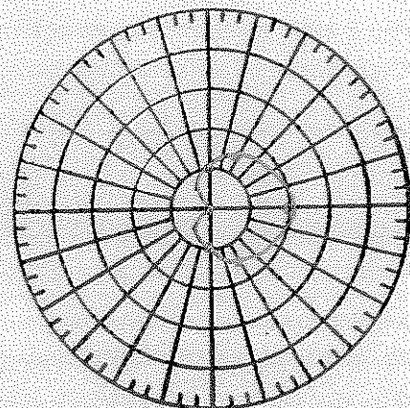
$$A = \frac{4\pi}{3} + \sqrt{3} - \frac{\pi}{3} - \sqrt{3}$$

$A = \pi$ sq. units

8. (21 POINTS) Consider the polar equation $r = 1 + \cos\theta$.

a. (5 POINTS) Sketch a graph of the polar equation by hand.

| $r$ | $\theta$ |
|---|---|
| $2$ | $0$ |
| $\frac{3}{2}$ | $\frac{\pi}{3}$ |
| $\frac{3}{2}$ | $\frac{5\pi}{3}$ |
| $1$ | $\frac{\pi}{2}$ |
| $1$ | $\frac{3\pi}{2}$ |
| $0$ | $\pi$ |



b. (10 POINTS) Find the points of

i. Horizontal tangency $y = r\sin\theta$

$r' = -\sin\theta$

$\frac{dy}{d\theta} = r\cos\theta + r'\sin\theta$

$\frac{dy}{d\theta} = (1+\cos\theta)\cos\theta - \sin\theta\sin\theta$

$\frac{dy}{d\theta} = \cos\theta + \cos^2\theta - \sin^2\theta$

$\frac{dy}{d\theta} = \cos^2\theta + \cos\theta - 1 + \cos^2\theta$

$0 = 2\cos^2\theta + \cos\theta - 1$

$0 = (2\cos\theta - 1)(\cos\theta + 1)$

$\cos\theta = \frac{1}{2}$ or $\cos\theta = -1$

$\theta = \frac{\pi}{3}, \frac{5\pi}{3}$ $\cancel{\theta = \pi}$

$\boxed{\left(\frac{3}{2}, \frac{\pi}{3}\right), \left(\frac{3}{2}, \frac{5\pi}{3}\right)}$

ii. Vertical tangency $x = r\cos\theta$

$\frac{dx}{d\theta} = -r\sin\theta + r'\cos\theta$

$\frac{dx}{d\theta} = -(1+\cos\theta)\sin\theta - \sin\theta\cos\theta$

$\frac{dx}{d\theta} = -\sin\theta - 2\sin\theta\cos\theta$

$0 = -\sin\theta(1 + 2\cos\theta)$

$\sin\theta = 0$ or $2\cos\theta = -1$

$\theta = 0, \cancel{\pi}$ $\quad \cos\theta = -\frac{1}{2}$

$\theta = \frac{2\pi}{3}, \frac{4\pi}{3}$

$\boxed{(2, 0), \left(\frac{1}{2}, \frac{2\pi}{3}\right), \left(\frac{1}{2}, \frac{4\pi}{3}\right)}$

c. (10 POINTS) Find the arc length over the interval from $[0, 2\pi]$.

Recall:

$1 + \cos\theta = 2\cos^2\left(\frac{\theta}{2}\right)$

$S = 2\int_0^{\pi} \sqrt{(1+\cos\theta)^2 + (-\sin\theta)^2}\, d\theta$

$S = 2\int_0^{\pi} \sqrt{1 + 2\cos\theta + \cos^2\theta + \sin^2\theta}\, d\theta$

$S = 2\int_0^{\pi} \sqrt{2(1+\cos\theta)}\, d\theta$

$S = 2\int_0^{\pi} \sqrt{2\left(2\cos^2\left(\frac{\theta}{2}\right)\right)}\, d\theta$

$S = 4\int_0^{\pi} \cos\left(\frac{\theta}{2}\right) d\theta$

$S = 4 \cdot 2\sin\frac{\theta}{2} \Big|_0^{\pi}$

$S = 8\left[\left(\sin\frac{\pi}{2} - \sin 0\right)\right]$

$S = 8(1 - 0)$

$\boxed{S = 8 \text{ units}}$